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Reconstruction Theorem for Groupoids and Principal Fiber Bundles

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The loop space formulation of 3 + 1 canonical quantum gravity premises that all physical information is contained within the holonomy loop functionals. This assumption is the result of the reconstruction theorem for a principal fiber bundle on a base loop space. The gauge connection for interacting gauge theories is more appropriately and readily reconstructed on a path space as opposed to a loop space. We generalize the reconstruction theorem to a base path space. Employing a holonomy groupoid map and a path connection, we trivially construct an abstract Lie groupoid from which a principal fiber bundle and gauge connection can be derived as distinctive examples. The groupoid reconstruction theorem is valid on both connected and nonconnected base manifolds, unlike the holonomy group reconstruction theorem, which can only be utilized for connected manifolds.

1. INTRODUCTION

Due to the intrinsic inadequacies of the perturbative program in constructing a quantum theory of gravity, nonperturbative programs are currently being explored instead (Ashtekar, 1991). A significant development in canonical nonperturbative quantum gravity, which leads to a simplification of the classical constraint equations, utilizes spinorial variables and is referred to as Ashtekar's general relativity (Ashtekar, 1986). A further progression of this approach led Rovelli and Smolin (1988, 1990) to the loop representation of quantum gravity. Using the loop representation, they demonstrated that solutions to the Hamiltonian constraint in 3 + 1 canonical quantum gravity exist if the constraints are expressed in terms of loop representation variables. These solutions are solutions in knot classes.

Many authors have applied the loop representation to various physical models, such as lattice gauge theory (Loll, 1992, 1993; Witten, 1989), Yang–

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Mills gauge theory (Loll, 1991), linearized gravity (Ashtekar *et al.*, 1991; Zegwaard, 1992, 1993), and quantum general relativity (Zegwaard, 1991; Bengtsson, 1989; Rayner, 1990; Gambini, 1991), to name but a few. However, as yet, the loop space approach to quantum gravity has not produced a physically meaningful quantum observable.

In this paper we introduce a modification of the reconstruction theorem that allows us to propose an alternative to the Wilson loop (Wilson, 1974) which appears in conventional loop representations approaches. The alternative functional is gauge groupoid invariant (Brown, 1987) and exists on path space rather than loop space and has the form

$$P \exp \int_x^y A \tag{1}$$

In this paper we do not explicitly use (1) in a physical context, we merely illustrate mathematically that such a path-dependent holonomy can be used to reconstruct connections and principal fiber bundles. Path space is more general than loop space as it contains both paths and loops, the latter of which are a special case of the former. Moreover, in terms of a principal fiber bundle the structure group exists on loop space, while the bundle is defined on path space, which naturally has a Lie groupoid structure (Mayer, 1990). A Lie groupoid Ω is equivalent to a principal fiber bundle except that it avoids an arbitrary choice of base point, a by-product of which is that the groupoid does not have a single group structure, but a bundle group structure $\mathcal{G} = \{G_x\}_{x \in X}$ in which G_x is the group structure at x, where x is a point in the base manifold X (Mackenzie, 1989). The bundle group structure associated with Lie groupoids makes the reconstruction formalism we present in this paper particularly well suited to interacting gauge field theories which have a bundle group structure. Furthermore, because a base point need not be specified, the reconstruction theorem for groupoids can be defined on manifolds which are not connected as well as connected manifolds (Brown, 1987).

In this paper we show that a Lie groupoid Ω can be constructed from a holonomy groupoid map $\mathcal{H}_A$. The holonomy groupoid map $\mathcal{H}_A$ maps elements of ΠX (the fundamental groupoid, introduced in Section 3) into the bundle group structure $\mathcal{G}$. Furthermore, a path connection (Mackenzie, 1987) can be defined which lifts paths from the base space ΠX into the Lie groupoid Ω. This is similar to the reconstruction theorem (Anandan, 1983; Barrett, 1991), except that loop space is generalized to path space and a principal fiber bundle is generalized to a Lie groupoid. The reconstruction theorem is often quoted as holding only for loop space. We show in this paper that this is not the case. In recent years several modifications of the reconstruction theorem have appeared. In Wilkins (1991) it was shown that the length of the loops must be constrained in order to reconstruct a principal fiber bundle; in Hajac (1993) a simplification of Barrett's connection construction was proposed. The central modification we introduce in this paper is the employment of groupoids over groups. The holonomy groupoid map $\mathcal{H}_A$ can naturally generate a Lie groupoid Ω, due to its isomorphic nature, which can further be specialized to the principal fiber bundle Ω_x. In contrast, the holonomy group map, due to its automorphic nature, generates only the fibers of the bundle. This is an important point, as it illustrates one of the fundamental differences between groupoids and groups applicable to the reconstruction theorem. Loop space has been used in simple gauge theories, such as the free Maxwell field (Ashtekar and Rovelli, 1992). However, it can readily be demonstrated that the path space formalism is essential in order to construct a physically sensible holonomy-dependent quantum field theory (Wood, 1996).

2. LOOP SPACE

A loop γ is a smooth continuous mapping of the unit interval $I = [0, 1]$ into a (topological) space X such that $\gamma(0) = \gamma(1)$. The collection of all loops in X with a defined base point $* = \gamma(0) = \gamma(1)$ is called loop space and is denoted by ΩX. The loop group, denoted by LG, is constructed from loop space ΩX via a homotopy relation which quotients ΩX into equivalence homotopy classes. Homotopy on loop space is usually taken to be a thin homotopy as in Lewandowski (1993). However, as pointed out in Caetano and Picken (1994), thin homotopy has particular disadvantages in constructing the connection for the reconstruction theorem. In particular, the pullback of the connection, from a smooth map (the loop) back into the bundle, must be generalized to the pullback of the connection on a continuous piecewise smooth map (the composite paths of the loop). In order to avoid this unnecessary complication, Caetano and Picken (1994) define a weaker thin homotopy called intimate homotopy on path space. We shall utilize such homotopy in our reconstruction theorem in Section 3.

The reconstruction theorem in Barrett (1991) relies on thin homotopy. Two loops $\gamma, \gamma' \in \Omega X$ are thinly homotopic if there exists a map

$$h\colon \quad I \times I \to X \tag{2}$$

such that:

(1) $h(s, t) \in \Omega X$ for all s and $t \in I$.
(2) $h(0, t) = \gamma(t)$ and $h(1, t) = \gamma'(t)$ for all $t \in I$.

The concept of holonomy in the reconstruction theorem pertains to a holonomy group element. Geometrically the holonomy represents the parallel translation of a quantity along some curve

$$\gamma = \{\gamma(s)\colon s \in I\ \gamma(0) = x \text{ and } \gamma(1) = y\} \tag{3}$$

If a connection is introduced onto a principal bundle, then the holonomy is a solution to the parallel translation differential equation (Nash and Sen, 1989)

$$dH(x, y) = A(\gamma)H(x, y) \tag{4}$$

with the boundary condition $H(x, x)$ equal to the identity element of the structure group G. The solution to (5) can be formally expressed as

$$H = P \exp \int_x^y A \tag{5}$$

where P denotes path ordering and results from the path γ being decomposed into n smaller paths. The path ordering in the above equation takes the form

$$H = 1 + \sum_{n=1}^{\infty} g^n \int A_{\mu_n} \cdots A_{\mu_1} \, d\gamma^{\mu_n} \cdots d\gamma^{\mu_1} \tag{6}$$

where γ^{μ_i} represents a smaller composite path of γ^{μ} for i an arbitrary element of n. The holonomy group, denoted by HG, is formed from the collection of holonomies H along closed paths (loops) γ, where $\gamma(0) = \gamma(1)$. The holonomy map H_A is a map

$$H_A\colon \quad \Omega X \to G \tag{7}$$

with a composition map $H_A(\gamma).H_A(\gamma') = H_A(\gamma'.\gamma)$ and inverse map $H_A^{-1}(\gamma) = H_A(\gamma^{-1})$ defined. We are now in a position to define the reconstruction theorem for loop space (Barrett, 1991) in terms of such a map as follows.

Consider a manifold M which is connected and paracompact and upon which a base point $*$ is defined such that a loop space ΩX may be constructed. Furthermore, assume there exists a map H_A: $\Omega X \to G$ which satisfies the following three properties:

(H1) Under the composition of loops, H_A is a homomorphism

$$H_A(\gamma.\gamma') = H_A(\gamma')H_A(\gamma) \tag{8}$$

(H2) For loops which differ by a reparametrization homomorphism ϕ: $I \to [a, b] \subseteq I$, we have

$$H_A(\gamma) = H_A(\phi(\gamma)) \tag{9}$$

(H3) Consider a finite-dimensional set of smooth loops $\{\gamma\}$: $U \to \Omega X$, where U is an open subset of R^n for any n. The composition of the map with this set

$$H_A(\{\gamma\})\colon \quad U \to G \tag{10}$$

is smooth.

If the conditions (H1)–(H3) are satisfied, there will exist a principal bundle $P = P(X, G)$, a point $b \in \pi^{-1}(*)$, and a connection Γ on P such that H_A is the holonomy map of the bundle.

3. PATH SPACE AND THE HOLONOMY GROUPOID

In order to illustrate that the reconstruction theorem can be generalized to hold for path space, we must develop the path space analogy of the holonomy map H_A. First we define path space formally. On a path connected space X with a base point $*$, the collection of all paths c: $I \rightarrow PX$ in X originating at the point $* = c(0)$ is called path space and is denoted PX. The Serre fibration of path space forms the triple (PX, p, X), where p is the map p: $PX \rightarrow X$. A typical fiber of (PX, p, X) is the loop space ΩX.

In this paper we use a rank-one homotopy to construct the fundamental groupoid ΠX from path space PX. We use the rank-one intimate path homotopy as defined by Caetano and Picken (1994). Two paths c, c': $I \rightarrow PX$ are called intimate homotopic if there exists a map

$$h: \quad I \times I \rightarrow X \tag{11}$$

such that:

(1) h is smooth on $I \times I$.

(2) The rank of the homotopy must be equal to or less than one. The rank of the homotopy is determined by calculating the rank of the homotopy Jacobian Dh, so

$$\mathrm{rank}(Dh_{(s,t)}) \leq 1 \qquad \text{for all} \quad (s, t) \in I \times I \tag{12}$$

(3) There exists an ϵ, where $0 < \epsilon < 1/2$, such that

$$\begin{aligned} 0 \leq s \leq \epsilon, &\qquad h(s, t) = c(t) \\ 1 - \epsilon \leq s \leq 1, &\qquad h(s, t) = c'(t) \\ 0 \leq t \leq \epsilon, &\qquad h(s, t) = c(0) \\ 1 - \epsilon \leq t \leq 1, &\qquad h(s, t) = c(1) \end{aligned} \tag{13}$$

When two paths are intimate homotopic equivalent, which we denote by $c \sim c'$, composition is then defined, implying that $c(1) = c'(0)$.

Thin homotopy quotients the loop space ΩX into the loop group LG; similarly, path homotopy will quotient the path space PX into the path groupoid, often called the fundamental groupoid. The fundamental groupoid, denoted by ΠX, is composed of objects which represent path homotopy equivalence classes. That is, if two paths are homotopic, then they belong to the same homotopy equivalence class. A groupoid is a more general object than the group; for example, a groupoid with just one object is a group. In the Appendix the salient features of groupoids are reviewed.

We have already introduced the holonomy on a open path in equation (5). The holonomy groupoid, denoted by $\mathcal{HG}$, is constructed from the collection of all such holonomies. In the next section we will show that the holonomy groupoid associated with a path connection Λ (to be defined in Section 4) is equal to the collection of endpoints of the "lifted" paths. That is,

$$\mathcal{HG} = \{\Lambda(\beta(c)): \quad c: I \to \Pi X\} \tag{14}$$

For β notation see Appendix. The holonomy groupoid map $\mathcal{H}_A$ is a map

$$\mathcal{H}_A: \quad \Pi X \to \mathcal{G} \tag{15}$$

such that groupoid composition is satisfied, that is, $\mathcal{H}_A(c).\mathcal{H}_A(c') = \mathcal{H}_A(c.c')$ if $c(1) = c'(0)$, and an inverse exists $\mathcal{H}_A^{-1}(c) = \mathcal{H}_A(c^{-1})$.

A fundamental property of the holonomy (Kobayashi and Nomizu, 1963) states that for any two distinct elements of the holonomy group, say $H(x)$ and $H(y)$, there will always exist an isomorphic "relation" between $H(x)$ and $H(y)$ as follows:

$$\mathcal{H}: \quad H(x) \to H(y), \qquad \text{where} \quad H(y) = \tau_c H(x) \tau_c^{-1} \tag{16}$$

where τ_c is the parallel translation along the path $c = \{c(s): s \in I, c(0) = x \text{ and } c(1) = y\}$ (Bergery and Ikemakhen, 1993). Note that such an isomorphic map corresponds to an element of the holonomy groupoid.

4. PATH CONNECTION

A connection Γ in a principal fiber bundle P is a decomposition of the tangent space in P at a point $u \in P$ into a direct sum of horizontal and vertical tangent subspaces,

$$T_u(P) = H_u(P) + V_u(P) \tag{17}$$

such that the right action R of the structure group G commutes with the horizontal subspace H_u,

$$H_{R_g u} = R_g H_u \text{ for } g \in G \tag{18}$$

A 1-form can be associated with the connection Γ on a principal bundle. The 1-form takes values in the Lie algebra of the structure group G. In this paper we use a path connection rather than a connection in order to avoid the necessity of "manually" lifting paths from X into the Lie groupoid Ω. A path connection has two major advantages over the infinitesimal connection. First, by definition it "automatically" lifts paths into Ω. Second, the lifted paths are reparametrization invariant. The path connection was first introduced by Bishop and Crittenden (1964) and Singer and Thorpe (1967). It was first defined in the context of groupoids by Virsik (1971). In this section we shall summarize the results we need from the theory of path connections (Mackenzie, 1987) in order to reproduce our generalization of the reconstruction theorem. We use a C^∞-path connection (Mackenzie, 1987) to form a holonomy groupoid map $\mathcal{H}_A$ for path space. A C^∞-path connection in a Lie groupoid Ω on a base space ΠX is a map

$$\Lambda: \quad \Pi X \to P_{\Omega_x^x}(\Omega) \tag{19}$$

where $P_{\Omega_x^x}(\Omega)$ are the set of piecewise paths in Ω that start at an identity Ω_x^x of Ω. The path connection Λ satisfies the following properties:

(1) The lift Λ of a path c into the bundle must start at an identity element Ω_x^x [where $x = c(0)$] in Ω. That is,

$$\Lambda(c)(0) = \Lambda(c)(0)) \tag{20}$$

The projection β: $\Omega \to \Pi X$ of a path $\Lambda(c)$ in the Lie groupoid Ω onto the base space ΠX is c. That is,

$$\beta(\Lambda(c)) = c \tag{21}$$

(2) When the lifted path $\Lambda(c)$ is reparametrized via the homeomorphism ϕ: $I \to [a, b] \subseteq I$ to the path $\Lambda(\phi(c))$, the reparametrized path no longer starts at the identity point Ω_x^x. To rectify this, we introduce R, which translates the reparametrization of the lift to the identity element in the same fiber over $c(0)$,

$$\Lambda(c(\phi)) = R_{\Lambda^{-1}(c)(\phi(0))}((\Lambda(c)(\phi))) \tag{22}$$

where $\Lambda^{-1}(c)(\phi(0))$ is the projection of $\Lambda(c)(\phi(0))$ into the base and is equal to $c(\phi)(0)$.

Three important properties which arise from the above conditions on the path connection Λ are: (i)

$$\Lambda(c_x) = c_{\Lambda(x)} \tag{23}$$

where c_x is the constant path at x and $\Lambda(x)$ is the point x lifted into the bundle; (ii)

$$\Lambda(c(1-t)) = \Lambda^{-1}(c), \qquad \text{where} \quad c(1-t) = c^{-1}(t) \quad \text{for} \quad t \in I \tag{24}$$

and (iii)

$$\Lambda(cc') = \Lambda(c)\Lambda(c') \tag{25}$$

Observe the similarity between these properties and the properties of the holonomy map.

The path connection Λ is an element of the Lie groupoid Ω and as such inherits a differential structure from the Lie groupoid. This differential structure satisfies the following three conditions (Mackenzie, 1987).

(i) If c is differentiable at $t_0 \in I$, then $\Lambda(c)$ is differentiable at t_0.

(ii) For $c, c'\colon I \to X$ we have

$$\frac{dc}{dt}(t_0) = \frac{dc'}{dt}(t_0) \qquad \text{for some} \quad t_0 \in I \tag{26}$$

Then this implies that

$$\frac{d\Lambda(c)}{dt}(t_0) = \frac{d\Omega(c')}{dt}(t_0) \tag{27}$$

(iii) For $c, c', c''\colon I \to X$ we have

$$\frac{dc}{dt}(t_0) + \frac{dc'}{dt}(t_0) = \frac{dc''}{dt}(t_0) \qquad \text{for some} \quad t_0 \in I \tag{28}$$

which implies that

$$\frac{d\Lambda(c)}{dt}(t_0) + \frac{d\Lambda(c')}{dt}(t_0) = \frac{d\Lambda(c'')}{dt}(t_0) \tag{29}$$

5. RECONSTRUCTION THEOREM AND PATH SPACE

The groupoid formalism of reconstruction presented in this paper has several advantages over other reconstruction theorems. Consider, for example, an interacting field theory in which the interacting particles $1, 2, \ldots, n$ have associated group structures $G1, G2, \ldots, Gn$. To reconstruct the associated bundle $P(G1, G2, \ldots, Gn)$ the conventional reconstruction theorem must be applied several times. In contrast, the groupoid reconstruction theorem need only be applied once, as it naturally accommodates the multiple symmetries associated with interacting field theories.

Another advantage of the groupoid approach is that the base manifold need not be connected, as it is unnecessary to preassign a fixed base point. As a result, the groupoid method presented in this paper is simpler to utilize and has greater application. It is mathematically geared to describe theories involving multiple symmetries and furthermore holds for manifolds which are not connected. A detailed application of the groupoid within the context of interacting field theories is presented in Wood (1996).

In this section we further illustrate how the conventional reconstruction theorem can be obtained as a special example of our more general reconstruction theorem. By selecting a fixed base point and restricting the base manifold to be connected, we can obtain the reconstruction theorem based on loop space. Furthermore, motivated by the utility of groupoids, we replace the loops by paths; the advantage in doing so is that we can utilize intimate homotopy introduced in Section 3 to automatically invest the holonomy group map with the property of piecewise smoothness, i.e., the property H3.

Let us define a map $\mathcal{H}_A: \Pi X \to \mathcal{G}$ which maps between the fundamental groupoid ΠX (acting as the base manifold space) and the bundle group structure $\mathcal{G}$ over ΠX. First we check that the holonomy map is homomorphic under composition (that is, property H1).

(H1) $\mathcal{H}_A(c.c') = \mathcal{H}_A(c)\mathcal{H}_A(c')$.

Proof. Groupoid composition denoted by $c.c'$ is defined only if $\beta(c) = \alpha(c')$ in ΠX (for β and α notation see Appendix). From the definition of the holonomy groupoid (14),

$$\mathcal{H}_A(c).\mathcal{H}_A(c') = \Lambda(\beta(c)).\Lambda(\beta(c')) \qquad \text{for all} \quad c, c': I \to X$$

$$= \Lambda(\beta(c'.c)) \tag{30}$$

$$= \mathcal{H}_A(c.c') \tag{31}$$

by (25). So property H1 is satisfied.

The path connection $\Lambda(c)$ is reparametrization invariant by construction and so the holonomy groupoid $\mathcal{HG}$ is also reparametrization invariant. We denote this property (H2),

$$\mathcal{H}_A(c^{-1}) = \mathcal{H}_A^{-1}(c)$$

which is a consequence of H1 and reparametrization.

(H3) The third property is a smoothness condition. For a smooth, finite-dimensional family of paths $\{c\}: U \to \Pi X$ with U an open subset of R^n, for any n, the composition map $\mathcal{H}_A(\{c\}): U \to \Pi X \to G$ is piecewise smooth. Unlike loop space, path space PX is piecewise smooth and so obviously $\mathcal{H}_A(\{c\})$ is automatically piecewise smooth. In the reconstruction theorem for loop space (Barrett, 1991), where loop space is smooth, it is necessary to show that the holonomy map is piecewise smooth, rather than just smooth. This illustrates that the path formulation leads to a simplification of the reconstruction theorem.

We utilize a special feature of the path connection Λ on the fundamental groupoid ΠX (Mackenzie, 1987), namely, there is only one unique path connection which acts on ΠX. In such circumstances the holonomy groupoid $\mathcal{H}_A$ reproduces the whole Lie groupoid Ω, i.e., $\mathcal{H}_A: \Pi X \to \Omega$. As such, a holonomy groupoid map on the fundamental groupoid entirely reconstructs the whole Lie groupoid.

5.1. Bundle Construction

Given a holonomy groupoid map, we can automatically construct a Lie groupoid Ω. To reconstruct a principal fiber bundle Ω_x from Ω, we must restrict the base manifold to be connected and must define a base point, say x. The principal fiber bundle Ω_x is a Lie groupoid with a fibered point over which the group structure is defined; hence we must constrain the holonomy groupoid map as follows:

$$\mathcal{H}_A|_x\colon \quad \Pi X|_x \to G|_x \tag{32}$$

where $\Pi X|_x = \Pi_1 X(x, X)$, the set of automorphic maps at x called the fundamental group, and $\mathcal{G}|_x = G_x$, the structure group at x. As such, $\mathcal{H}_A|_x = H_A$ is the holonomy group map at x. Such a restriction of the holonomy groupoid reconstruction is analogous to the conventional reconstruction theorem on loop space, except that we utilize intimate homotopy, which simplifies the theorem, as $\mathcal{H}_A|_x$ is automatically piecewise smooth. Utilizing this feature, we now proceed to show that $\mathcal{H}_A|_x$ will reconstruct a principal fiber bundle in detail.

The minimum information required to construct a principal bundle is the base manifold, in our case ΠX, an open set on ΠX, say $\{U_i\}$, the transition functions $t_{ij}(u)$, and the structure group G_x (which is isomorphic to the fiber space F for a principal bundle). We construct the bundle space

$$E = \bigcup_i (U_i \times G_x)/\sim$$

where $\sim$ is an equivalence intimate homotopy relation on the bundle space.

We will use the map $\mathcal{H}_A$ and its properties (H1)–(H3) to construct a bundle space E. To specify a principal fiber bundle, we require a base point defined on the base manifold, say $x = c'(1)$. Then for g and g' elements of the fiber above $c'(1)$, the transition functions relating g to g' are $t_{ij}(c'(1))$. The transition functions t_{ij} can be represented by the map $\mathcal{H}_A|_x$, explicitly $t_{ij}(c'(1)) = \mathcal{H}_A|_x(c'^{-1}.c)$. Consider U_i and U_j as open subsets of ΠX. An intimate homotopy equivalence relation, denoted $\sim$, between $(c, g) \sim (c', g')$, where $(c, g) \in U_i \times G_x$ and $(c', g') \in U_j \times G_x$, will exist iff

$$c \sim c' \qquad \text{and} \qquad g = t_{ij}(c'(1))g' \tag{33}$$

Such a relation is an equivalence only if the properties of reflexivity, symmetry, and transitivity hold.

Proof. Consider an arbitrary element of E, i.e., (c, g). First we show the reflexivity property $(c, g) \sim (c, g)$: Obviously, $c \sim x$, as

$$\begin{aligned} g &= t_{ij}(c(1))g \\ &= \mathcal{H}_A|_x(c^{-1}.c)g \\ &= g \end{aligned} \tag{34}$$

then reflexivity holds. Because c and c are obviously in the same equivalence class $[c]$, the only path which will satisfy the partial composition law of groupoids is $[c^{-1}]$.

The symmetry property $(c, g) \sim (c', g')$, then $(c', g') \sim (c, g)$, where $c, c' \in [c]$: If $(c, g) \sim (c', g')$, the equivalence relation requires that $c \sim c'$ and $g = \mathcal{H}_A|_x(c'^{-1}.c)g'$. Therefore by (H1) and (H2)

$$\begin{aligned} g' &= \mathcal{H}_A|_x^{-1}(c'^{-1}.c)g \\ &= \mathcal{H}_A|_x(c^{-1}.c')g \end{aligned} \tag{35}$$

This completes the demonstration of the symmetry property.

Finally, we show that the transitivity property, $(c, g) \sim (c', g')$ and $(c', g') \sim (c'', g'')$ implies $(c, g) \sim (c'', g'')$, holds.

Now $(c, g) \sim (c', g')$ implies $c \sim c'$ and $g = \mathcal{H}_A|_x(c'^{-1}.c)g'$, while $(c', g') \sim (c'', g'')$ implies $c' \sim c''$ and $g' = \mathcal{H}_A|_x(c''^{-1}.c')g''$. So

$$\begin{aligned} g'' &= \mathcal{H}_A|_x^{-1}(c''^{-1}.c').\mathcal{H}_A|_x^{-1}(c'^{-1}.c)g \\ &= \mathcal{H}_A|_x(c''^{-1}.c)g \quad \text{by (H1) and (H2)} \end{aligned} \tag{36}$$

and thus transitivity holds. Thus we have shown that $\sim$ is a homotopy equivalence relation quotienting $\Pi X \times G_x$ via the use of the map $\mathcal{H}_A$ restricted to $\mathcal{H}_A|_x$ and its associated properties. In Section 5.3 we will show that the map $\mathcal{H}_A|_x$ is the holonomy groupoid map.

5.2. Connection Construction

The bundle so far constructed inherents a differentiable structure from the Lie groupoid Ω associated with $\mathcal{H}_A|_x$. Define a map $\Lambda|_x$ on the bundle Ω_x as follows:

$$\Lambda|_x\colon \quad \Pi X|_x \to P_{\Omega_x^x}(\Omega)|_x = P_{\Omega_x^x}(\Omega_x) \tag{37}$$

We need to show that this map also inherits a differential structure and thus satisfies the properties (20)–(22) of a path connection.

Proof. (1) To show that (20) and (21) are satisfied, we utilize the properties of the α and β maps in the Appendix. Now $c(0) = \alpha(c)$ and so

$$\begin{aligned} \Lambda(\alpha(c)) &= \alpha(\Lambda(c)) \\ &= \Lambda(c)(0) \end{aligned} \tag{38}$$

In (21) note that while Λ lifts the path c, the β map projects the path. And so the Λ map cancels out the β map and hence the property follows.

(2) To show that (22) is satisfied, notice that $\Lambda(c(\phi))$: $[a, b] \to c_a$, where c_a is a path in the bundle starting at the point a. But by the definition of the lift $\Lambda(c)$ of a path c should start at $\Lambda(c(0))$. So the role of $R_{\Lambda^{-1}(c)(\phi(0))}$ is to automatically redefine $\Lambda(c(\phi))$ to $\Lambda(c(0))$.

5.3. Determining the Holonomy

We need to show that the map $\mathcal{H}_A|_x$ on a principal bundle Ω_x with path connection $\Lambda|_x$ is the holonomy group map, that is, an element of the holonomy group HG. In other words, we must show that

$$\begin{aligned} \mathcal{H}_A|_x(c) &= HG(c) \\ &= \Lambda|_x(\beta(c)), \qquad \text{where} \quad \beta(c) = \alpha(c) \end{aligned} \tag{39}$$

Proof. By definition, the holonomy groupoid consists of the endpoint values $\Lambda(\beta(c))$ of the lifted paths $\Lambda(c)$. The point $\Lambda(\beta(c))$ in the bundle space is on the bundle over $\beta(c) \in \Pi X$ and so is an element of the bundle structure group at $\beta(c)$, say $g \in G_{\beta(c)}$. If further we restrict the bundle space over $\beta(c)$ to the point x, then $\mathcal{H}\mathcal{G}|_x(c) = \Lambda|_x(\beta(c)) = g$, where $g \in G_{\beta(c)}|_x$ is a point on the fiber on the principal fiber bundle.

The holonomy groupoid map $\mathcal{H}_A$: $\Pi X \to \mathcal{G}$ acts on the path c: $I \to \Pi X$. The initial point of the path c is fixed to be the identity element of the bundle at $\alpha(c)$. And so $\mathcal{H}_A(c) \in \mathcal{G}$ and refers to the path's c endpoint $\beta(c)$ lifted into the bundle space; thus $\mathcal{H}_A|_x(c) = \Lambda|_x(\beta(c))$.

6. CONCLUSION

We have demonstrated that the reconstruction theorem can be generalized to reconstruct a Lie groupoid from which the special case of a principal fiber bundle can readily be derived. As such, we conclude that the conventional reconstruction theorem is a special case of the reconstruction theorem we have presented in this paper. The main advantages of the reconstruction theorem for paths presented in this paper over the reconstruction theorem for loops are that:

(1) Interacting field theories are more appropriately modeled.
(2) The base manifold need not necessarily be connected.
(3) The reconstruction theorem for loops can be recast in such a fashion that its associated homotopy is piecewise smooth as opposed to just smooth.

APPENDIX

There are many excellent reviews of groupoids; see, for example, Brown (1987), Mayer (1990), or Mackenzie (1987). In this Appendix the features of groupoids used in this paper are summarized.

A groupoid Ω over a base X is a set (Φ, X) where Φ are the set of isomorphic maps. Φ are the "elements" of the groupoid Ω. In the literature Φ is sometimes called the groupoid; however, in the present paper the groupoid will be denoted by Ω. X is a set of objects.

The groupoid has two projection maps α, β: $\Phi \to X$ called respectively the source and target maps, and one object inclusion map ϵ: $X \to \Phi$. The main difference between groups and groupoids is that only a partial composition is defined for groupoids, that is,

$$z_1.z_2: \quad \Phi.\Phi \to \Phi \qquad \text{for} \quad z_1 \text{ and } z_2 \text{ elements of } \Phi$$

The composition satisfies the following conditions:

(1) $\Phi.\Phi = \{z_1.z_2 \in \Phi.\Phi \text{ such that } \beta(z_1) = \alpha(z_2)\}$.
(2) If $\beta(z_1) = \alpha(z_2)$ and $\beta(z_2) = \alpha(z_3)$; then $z_1.(z_2.z_3) = (z_1.z_2).z_3$.
(3) $\alpha(\epsilon) = \beta(\epsilon)$ is the identity element of Ω.
(4) Each $z \in \Phi$ has an inverse z^{-1} such that $\alpha(z^{-1}) = \beta(z)$ and $\beta(z^{-1}) = \alpha(z)$, and $z.z^{-1} = \epsilon(\alpha(z))$ and $z^{-1}.z = \epsilon(\beta(z))$.

From this general definition we now consider the following groupoids relevant to our path space formulation.

Consider a groupoid Ω, with fixed point $x \in X$, that is, $\Omega_x = \{z \in \Phi$: such that $\alpha(z) = x\}$. Now $\{\Omega_x\}_{x \in X}$, which denotes the collection of all Ω_x as x varies over the base X, is a principal fiber bundle. The Ω_x are the fibers which are isomorphic to the structure group. Thus a principal fiber bundle is actually a groupoid with a fibered point over which the group structure is defined. The bundle projection map, which is usually denoted by π, is the α map.

Consider a groupoid Ω with two fixed points x and $y \in X$, that is, $\Omega_x^y = \{z \in \Phi$: where $\alpha(z) = x$ and $\beta(z) = y\}$. One example of a groupoid with this structure is that associated with a path homotopy set, often called the fundamental groupoid ΠX.

The holonomy group, sometimes called the vertex group or isotropy group, can also be constructed from a groupoid Ω, as follows: $\Omega_x^x = \{z \in \Phi$: such that $\alpha(z) = x = \beta(z)\}$ for $x \in X$ a fixed point. It is a groupoid with one element and hence has a group structure.

It will be important in our construction of the reconstruction theorem to note that Ω_x^y is more fundamental than Ω_x^x; Ω_x^x can be constructed from Ω_x^y as follows:

$$\Omega_x^x([\gamma]) = \Omega_x^y([c]).\Omega_c^y([c]^{-1}) = \Omega_x^y([c]).\Omega_y^x([c])$$

where $[\gamma]$ is an equivalence class of closed paths in path space ΠX and $[c]$ denotes an equivalence class of open paths in ΠX, with the inverse defined by $c^{-1}(t) = c(1 - t)$ for $t \in I$. The equivalence class $[\gamma]$ is related to $[c]$ by $[\gamma] = [c.c^{-1}] = [c].[c^{-1}]$.

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